Discrete Mathematics 65 (1987) 289–300
North-Holland

FRAMES FOR KIRKMAN TRIPLE SYSTEMS

D.R. STINSON
Department of Computer Science, University of Manitoba, Winnipeg, Canada R3T 2N2

Received 28 May 1985
Revised 13 July 1986

We introduce the idea of a 'frame' for a Kirkman triple system: loosely speaking, a frame is a Kirkman system with 'holes'. We construct many frames for Kirkman triple systems. We then apply these frames to the construction of Kirkman triple systems with subsystems. We construct a system of order v containing a system of order w, for all admissable $v \geq 4w-9$, with two exceptions.

1. Introduction

Let X be a finite set of elements called *points*. A *block* will be a subset of X of size at least 2. A *parallel class* is a partition of X into blocks. A *Kirkman triple system* (or KTS) on X is a set $\mathcal{P}$ of parallel classes of blocks of size 3, in which any unordered pair of points is contained in a unique block. It follows that the number of parallel classes is $\frac{1}{2}(|X|-1)$; hence $|X|$ is odd. Also $|X|$ must be divisible by 3, so $|X| \equiv 3 \pmod 6$. We say that $|X|$ is the order of the KTS, and denote a Kirkman triple system of order v by KTS(v).

In 1970, Ray-Chaudhuri and Wilson [7] proved

Theorem 1.1. *There exists a Kirkman triple system of order v if and only if $v \equiv 3 \pmod 6$.*

This paper is a study of a special type of generalization of KTS, which we call a *Kirkman frame*, or frame. Intuitively, a frame is a KTS with 'holes'. We give a formal definition.

Let X be a finite set, and let $\mathcal{G}$ be a partition of X (the members of $\mathcal{G}$ will be called *holes*). A *$\mathcal{G}$-frame* is a set $\mathcal{P}$ of partial parallel classes of X, which satisfies the following conditions:

(1) Each $P \in \mathcal{P}$ is a partition of $X \setminus G$, for some $G \in \mathcal{G}$;

(2) The unordered pairs contained in the blocks in $\mathcal{P}$ are precisely those which come from different holes of $\mathcal{G}$, each such pair occurring exactly once.

The *type* of the $\mathcal{G}$-frame will be the multiset $\{|G| : G \in \mathcal{G}\}$. We use the notation $1^{u_1}2^{u_2}\cdots$ to describe a multiset containing u_1 1's, u_2 2's, etc.

Certainly $|X \setminus G| \equiv 0 \pmod 3$, for all $G \in \mathcal{G}$. It is also necessary that

0012-365X/87/$3.50

$|G| \equiv 0 \pmod 2$ for all $G \in \mathcal{G}$. This follows as a consequence of the following result.

Theorem 1.2. *If $\mathcal{P}$ is a $\mathcal{G}$-frame, then, for any $G \in \mathcal{G}$, there are precisely $\frac{1}{2}|G|$ partial parallel classes in $\mathcal{P}$ which partition $X \setminus G$.*

Proof. Denote by $c(G)$ the number of partial parallel classes partitioning $X \setminus G$. Then the total number of blocks

$$b = \sum_{G \in \mathcal{G}} \tfrac{1}{3} c(G)(|X| - |G|).$$

Now a point $x \in G$ occurs in precisely $r_x = \frac{1}{2}(|X| - |G|)$ blocks, so

$$b = \sum_{G \in \mathcal{G}} \tfrac{1}{6} |G| (|X| - |G|).$$

Hence,

$$0 = \sum_{G \in \mathcal{G}} (|X| - |G|)(2c(G) - |G|).$$

If we define $c = \sum_{G \in \mathcal{G}} c(G)$, then, for any $x \in G$, $r_x = c - c(G)$. Hence, $c(G) - \frac{1}{2}|G| = c - \frac{1}{2}|X|$, for all $G \in \mathcal{G}$. Now, we obtain

$$0 = \sum_{G \in \mathcal{G}} (|X| - |G|)(2c(G) - |G|) = \sum_{G \in \mathcal{G}} (|X| - |G|)(2c - |X|),$$

which implies $2c = |X|$. Hence, for any $G \in \mathcal{G}$,

$$c(G) = c - \tfrac{1}{2}|X| + \tfrac{1}{2}|G| = \tfrac{1}{2}|G|,$$

as desired. □

In this paper we describe some constructions for frames, and we determine necessary and sufficient conditions for the existence of a frame of type t^u (such a frame is called a *uniform*). The requirement is that t be even and $t(u-1) \equiv 0 \pmod 3$.

As an application of frames, we investigate the construction of KTS with subsystems. We prove that for all $v, w \equiv 3 \pmod 6$, $v \geq 4w - 9$, $(v, w) \neq (81, 15)$ or $(87, 21)$, there exists a KTS(v) containing a KTS(w) as a subsystem.

Applications of Kirkman frames to the construction of other types of designs are given in [1] and [8]. Also, we note that the constructions for nearly Kirkman triple systems given in [2] can be rephrased as frame constructions.

2. Terminology

We need first to introduce some terminology which will be used in this sequel.

A *pairwise balanced design* (or PBD) is a pair $(X, \mathcal{A})$, in which X is a set of points and $\mathcal{A}$ is a set of blocks, such that every unordered pair of points is contained in a unique block. We say that $(X, \mathcal{A})$ is a (v, K)-PBD if $|X| = v$ and $|A| \in K$, for all $A \in \mathcal{A}$. A set K of integers is called PBD-*closed* if $v \in K$ whenever there exists a (v, K)-PBD.

A *group-divisible design* (or GDD) is a triple $(X, \mathcal{G}, \mathcal{A})$, where X is a set of points, $\mathcal{G}$ is a partition of X into subsets called *groups*, and $\mathcal{A}$ is a set of blocks, such that a group and a block contain at most one common point, and every unordered pair of points from distinct groups is contained in a unique block.

A *transversal design* TD(k, n) is a GDD with kn points, consisting of k groups of size n, and n^2 blocks of size k. It is well-known that a TD(k, n) is equivalent to $k - 2$ mutually orthogonal Latin squares (MOLS) of order n.

A group-divisible design is *resolvable* if the set of blocks can be partitioned into parallel classses. A resolvable TD$(k - 1, n)$ is equivalent to a TD(k, n).

3. Constructions for frames

In the section we state three basic constructions for frames.

Construction 3.1. GDD construction

Let $(X, \mathcal{G}, \mathcal{A})$ be a GDD, and let $w: X \to \mathbb{Z}^+ \cup \{0\}$. (We say that w is a weighting). For each $A \in \mathcal{A}$, suppose there is a frame of type $\{w(x): x \in A\}$. Then there is a frame of type $\{\sum_{x \in G} w(x): G \in \mathcal{G}\}$.

Construction 3.2. Inflation by TDs

Suppose there is a frame of type T, and suppose there is a resolvable TD$(3, n)$ (i.e., $n \neq 2$ or 6). Then there is a frame of type $\{nt: t \in T\}$.

Construction 3.3. Filling in holes

Suppose there is a frame of type $\{t_1, \ldots, t_n\}$, and let $\varepsilon \geq 0$. For $1 \leq i \leq n$, suppose there is a frame of type $T_i \cup \{\varepsilon\}$, where $\sum_{t \in T_i} t = t_i$. Then there is a frame of type $\bigcup_{i=1}^{n} T_i \cup \{\varepsilon\}$.

4. Uniform frames

In this section we determine necessary and sufficient conditions for the existence of frames of type t^u. The necessary conditions are: t is even, and $t(u - 1) \equiv 0 \pmod 3$. Hence $t \equiv 2$ or $4 \pmod 6$ implies $u \equiv 1 \pmod 3$; and $t \equiv 0 \pmod 6$ implies no restriction on u. It is also easy to see that $u \geq 4$ or $u \leq 1$ (frames with one hole exist trivially and henceforth we shall ignore them).

Define $T_u = \{t: \text{there is a frame of type } t^u\}$. Then we observe that T_u is

PBD-closed. For, let $(X, \mathscr{A})$ be a (v, T_u)-PBD. Apply the GDD construction to the GDD$(X, \{\{x\}: x \in X\}, \mathscr{A})$. We obtain a frame of type t^v; hence $v \in T_u$.

We first observe that deleting a point from a KTS$(v+1)$ produces a frame of type $2^{v/2}$, and conversely, a frame of type 2^u yields a KTS$(2u+1)$. Hence, by Theorem 1.1, there is a frame of type 2^u for all $u \equiv 1 \pmod 3$.

Next, we consider frames of type 4^u. In [9], it is observed that a frame of type 4^4 is equivalent to a pair of (incomplete) orthogonal Latin squares of order 6 which are missing a pair of orthogonal subsquares of order 2. These squares are exhibited by Horton in [6]. To construct a frame of type 4^u, where $u \equiv 1 \pmod 3$, $u > 4$, we apply Construction 3.1 to a GDD having u groups of size 2 and blocks of size 4 (these GDDs are constructed in [3]). We give every point weight 2, and fill in frames of type 2^4.

So, we have proved the following result.

Lemma 4.1. *For all* $u \equiv 1 \pmod 3$, *there exists a frame of type* 2^u *and a frame of type* 4^u.

We now consider frames of type 6^u. We have the following result.

Lemma 4.2. *For all* $u \geqslant 4$, $u \neq 6$, *there is a frame of type* 6^u.

Proof. We distinguish 2 cases:

Case 1. $u \equiv 0$ or $(1 \bmod 4)$.

There is a $(3u+1, \{4\})$-PBD (i.e., a BIBD with blocksize 4, on $3u+1$ points). Delete a point, obtaining a GDD with u groups of size 3, and blocks of size 4. Apply the GDD construction giving every point weight 2.

Case 2. $u \equiv 2$ or $3 \pmod 4$

The results of Brouwer [4] show that there is a $(3u+1, \{4, 7\})$-PBD which contains a unique block of size 7, for all these values $u \equiv 2$ or $3 \pmod 4$, $u \neq 3, 6$. Since we assume $u \geqslant 4$, there is a point which doesn't occur in the block of size 7. Deleting this point, we proceed as in Case 1. □

Lemma 4.3. *For all* $u \geqslant 4$, $u \neq 6$, *there is a frame of type* 12^u.

Proof. The proof is that of Lemma 4.2, mutatis mutandis, using weight 4 in the GDD construction. □

In order to complete the determination of T_2, T_4, T_6, and T_{12}, we need only construct frames of types 6^6 and 12^6. A frame of type 6^6 is constructed in Fig. 1. A frame of type 12^6 is constructed by applying Construction 3.1 to a GDD with

Points: $(((Z_5 \times \{1,2\}) \cup \{\infty\}) \times Z_3) \cup \{\alpha_1, \alpha_2, \alpha_3\}$.

Groups: $\{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2)\}$ mod$(5,-;-)$,
$((\infty,1), (\infty,2), (\infty,3), \alpha_1, \alpha_2, \alpha_3\}$.

Partial parallel classes:

$\{(\infty,1), (1,1,0), (3,1,2)\}$, $\{\alpha_3, (4,1,1), (2,2,0)\}$,
$\{(\infty,2), (4,2,1), (3,2,2)\}$, $\{(2,1,1), (3,1,1), (4,1,2)\}$,
$\{(\infty,0), (3,1,0), (2,2,1)\}$, $\{(1,2,2), (3,2,1), (4,2,2)\}$,
$\{\alpha_1, (1,1,1), (2,2,2)\}$, $\{(4,1,0), (2,1,0), (1,2,0)\}$,
$\{\alpha_2, (2,1,2), (1,2,1)\}$, $\{(3,2,0), (4,2,0), (1,1,2)\}$,
all mod$(5,-;3)$.

Fig. 1. A frame of type 6^6.

six groups of size 6, and blocks of size 4, giving every point weight 2 (this GDD is shown to exist in [3]). Hence, we obtain

Lemma 4.4. *There exists a frame of type 6^u or 12^u if and only if $u \geq 4$, and there exists a frame of type 2^u or 4^u if and only if $u \equiv 1 \pmod 3$, $u \geq 4$.*

It is now an easy matter to construct frames of type t^u for all other values of t.

Theorem 4.5. *There exists a frame of type t^u if and only if t is even, $u \geq 4$, and $t(u-1) \equiv 0 \pmod 3$.*

Proof. We have already handled $t = 2, 4, 6$, and 12, so we assume t is not one of these values.

If $t \equiv 0 \pmod 6$, then write $t = 6t_1$ or $12t_1$, where $t_1 \neq 2, 6$. Then apply Construction 3.2 with $n = t_1$.

If $t \equiv 2$ or $4 \pmod 6$, then write $t = 2t_1$, where $t_1 \neq 2, 6$, proceed as above. □

5. Subsystems of Kirkman triple systems

Suppose we have a KTS(v) on point-set X with parallel classes $\mathcal{P}$. Let $Y \subseteq X$. We say that Y induces a *subsystem* of the KTS(v) if no block in $\mathcal{P}$ meets Y in precisely 2 points, and $\mathcal{P}$ induces a KTS on the blocks which are contained in Y. If Y contains w points, we say that the KTS(v) contains a sub-KTS(w).

In this section we investigate the question: for which ordered pairs (v, w) does there exist a KTS(v) containing a sub-KTS(w)? It is clearly necessary that $v, w \equiv 3 \pmod 6$. Also, it is not difficult to show that $v \geq 3w$. We conjecture that these conditions are sufficient. We are unable to prove this conjecture, but we are able to come close: $v \geq 4w - 9$ is shown to be a sufficient condition, with two possible exceptions. We prove this result in the remainder of this section.

The proof makes essential use of frames, together with a "filling in holes" construction analogous to Construction 3.3.

Construction 5.1. Suppose there is a frame of type T.

(1) If there is a KTS$(2t+1)$ for every $t \in T$, then there is a KTS$(2v+1)$ $(v = \sum_{t\in T} t)$;

(2) If, for some ε, there exists a KTS$(t+\varepsilon)$ containing a sub-KTS(ε) for all $t \in T$, then there is a KTS$(v+\varepsilon)$ $(v = \sum_{t\in T} t)$ which contains a sub-KTS(ε), and for every $t \in T$, a sub-KTS$(t+\varepsilon)$.

We now obtain

Corollary 5.2. *For all* $v \equiv 3 \pmod 6$, $v \geq 27$, *there exists a* KTS(v) *containing a sub-*KTS(9).

Proof. For all $u \geq 4$, there is a frame of type 6^u. Apply Construction 5.1 with $\varepsilon = 3$. □

We now describe three GDD constructions.

Theorem 5.3. *Suppose* $(X, \mathcal{G}, \mathcal{A})$ is a GDD *in which every block has size at least* 4. *Then there is a* KTS$(6|X|+3)$, *which, for every* $G \in \mathcal{G}$, *contains a sub-*KTS$(6|G|+3)$.

Proof. Give every point weight 6, and apply Construction 3.1. This gives a frame of type $\{6|G| : G \in \mathcal{G}\}$. Now apply Construction 5.1 with $\varepsilon = 3$. □

Theorem 5.4. *Suppose* $(X, \mathcal{G}, \mathcal{A})$ *is a* GDD *in which every block has size at least* 4, *and every group has size at least* 3. *Then there is a* KTS$(6|X|+9)$, *which, for every* $G \in \mathcal{G}$, *contains a sub-*KTS$(6|G|+9)$.

Proof. As in Theorem 5.3, obtain a frame of type $\{6|G| : G \in \mathcal{G}\}$. Now apply Construction 5.1 with $\varepsilon = 9$, noting that $6|G|+9 \geq 27$, for all $G \in \mathcal{G}$. Corollary 5.2 provides the necessary KTS$(6|G|+9)$ containing sub-KTS(9). □

Theorem 5.5. *Let* $(X, \mathcal{G}, \mathcal{A})$ *be a* GDD *in which every block has size at least* 4, *except for one block, A. Suppose also that every group meeting A has size at least* 4. *Then there is a* KTS$(6|X|+3)$ *which contains a sub-*KTS$(6|A|+3)$, *and for all groups* $G \in \mathcal{G}$ *with* $G \cap A = \emptyset$, *a sub-*KTS$(6|G|+3)$.

Proof. This is a slight variation of Construction 3.1. Give every point weight 6, and for every block $A_1 \neq A$, fill in a frame of type $6^{|A_1|}$. Now add 3 new points. Fill in: a KTS($6|A|+3$); for all $G \in \mathcal{G}$ with $G \cap A = \emptyset$, a KTS($6|G|+3$); and for all $G \in \mathcal{G}$ with $G \cap A \neq \emptyset$, a KTS($6|G|+3$) omitting the blocks of a sub-KTS(9) (this is where we require $|G| \geq 4$). □

We now further specialize the above three constructions, obtaining four corollaries which will be used to prove the bulk of our main result.

Corollary 5.6. *Suppose there is a* TD(k, v), *with* $k \geq 5$. *Then for* $4v \leq u \leq kv$, *there is a* KTS($6u+3$) *which contains a sub*-KTS($6v+3$).

Proof. Apply Theorem 5.3, deleting points from all but four groups of the given TD. □

Corollary 5.7. *Let* $v \geq 0$, *and suppose there is a* TD(k, n), *where* $n \geq v$ *and* $k \geq 5$. *For* $4n+v \leq u \leq (k-1)n+v$, *there is a* KTS($6u+3$) *which contains a sub*-KTS($6v+3$).

Proof. Delete no points from the first four groups of the TD, delete $n-v$ points from the fifth group, and delete any points from the remaining groups. Apply Theorem 5.3. □

Corollary 5.8. *Suppose there is a* TD(k, $v+1$), *where* $k \geq 5$ *and* $v \geq 4$. *Then, for* $5v \leq u \leq k(v+1)-1$, *there is a* KTS($6u+3$) *which contains a sub*-KTS($6v+3$).

Proof. Let x be a point in the first group of the TD, and let B be any block through x. First, we may delete points in $B \setminus \{x\}$. For each of groups number six through k, we delete up to $v+1-4=v-3$ points, or all $v+1$ points. Finally, delete the point x. We can do this so that the number of points remaining, u, can take on any value from $5v$ to $k(v+1)-1$. The GDD has all groups of size at least 4, and all blocks of size at least 4, except possibly B. So we take as groups all the blocks which pass through x. Apply Theorem 5.3. □

Corollary 5.9. *Suppose there is a* TD(k, $v-1$), *where* $k \geq 5$. *Then, for* $4v \leq u \leq k(v-1)+1$, *there is a* KTS($6u+3$) *containing a sub*-KTS($6v+3$).

Proof. From each of groups five through k of the TD, delete up to $v-1-3=v-4$ points, or all $v-1$ points. The number of points remaining, u, can assume any value from $4v-1$ to $k(v-1)$. Now, apply Theorem 5.4. □

We are now in a position to prove a preliminary result. Let T_6 denote the following set of positive integers:

$$n \in T_6 \begin{cases} \text{if } n \geq 5 \text{ is a prime power,} \\ \text{or if } n \geq 53, \\ \text{or if } n = 12, 15, 21, 35, 36, 39, 40, 45, 46, 48, 50, \text{ or } 51. \end{cases}$$

For a positive integer n, $N(n)$ denotes the maximum number of mutually orthogonal Latin squares of order n (so there is a TD(k, n) if and only if $k \leq N(n) + 2$). From Brouwer's list of lower bounds for $N(n)$ [5], one may verify the following property:

$$\text{If } n \in T_6,\ n \geq 7, \text{ then there is an } n_1 \in T_6,\ n_1 > n, \text{ such that } 4n_1 \leq (N(n)+1)n + 1. \qquad (*)$$

Note that (*) fails for $n = 5$: we have $N(5) = 4$, $n_1 = 7$, $4n_1 = 28$, and $(N(n)+1)n + 1 = 26$.

Lemma 5.10. *Suppose $v \geq 2$, $n \in T_6$, $n \geq \max\{7, v\}$. Then, for all $u \geq 4n + v$, there is a* KTS$(6u+3)$ *containing a sub*-KTS$(6v+3)$.

Proof. This follows from repeatedly applying Corollary 5.7, with a sequence of values of $n \in T_6$. The property (*) ensures there will be no 'gaps'. □

Lemma 5.11. *Suppose $v \geq 7$, $v \in T_6$. Then, for all $u \geq 4v$ there is a* KTS$(6u+3)$ *containing a sub*-KTS$(6v+3)$.

Proof. From Corollary 5.6, we get the desired KTS, for u in the range $4v \leq u \leq kv$ ($k = N(v) + 2$). For higher values of u, Lemma 5.10 takes over. □

Lemma 5.11 covers all values of v except for those in the following Table 1. In this table, we indicate the bound produced by Lemma 5.10. So, for these values

Table 1
Applications of Lemma 5.10

v	n	$4n+v$	v	n	$4n+v$
2	7	30	24	25	124
3	7	31	26	27	134
4	7	32	28	29	144
6	7	34	30	31	154
10	11	54	33	35	173
14	15	74	34	35	174
18	19	94	38	39	194
20	21	104	42	43	214
22	23	114	44	45	224
			52	53	264

of v, we have to construct KTS$(6u+3)$ containing KTS$(6v+3)$ for $4v \leqslant u \leqslant 4n+v$.

For $v \geqslant 18$, we apply Corollary 5.6 with $k=5$, to handle $4v \leqslant u \leqslant 5v$. Now, for all these v except $v=33$, we apply Corollary 5.8 with $k=6$. This does all u up to $6v+5 \geqslant 4n+v$ in each case. For $v=33$, we apply Corollary 5.9 with $k=6$.

For $v=6$, 10, 14, apply Corollary 5.9 with $k=6$, to handle $4v \leqslant u \leqslant 6v-5$. For $v=6$, also apply Corollary 5.8 with $v=6$, which covers $5v \leqslant u \leqslant 6v+5$.

So we have only to consider $v=2$, 3, and 4. First, we do $v=4$. Corollary 5.6 ($k=5$) does $16 \leqslant u \leqslant 20$; and Corollary 5.8 ($k=6$) does $20 \leqslant u \leqslant 29$. We have yet to do $u=30$, 31. These are done as follows.

Lemma 5.12. *There exist* KTS(183) *and* KTS(189) *containing sub*-KTS(27).

Proof. Start with a resolvable (28, 4, 1)-BIBD and adjoint ε new points ($0 \leqslant \varepsilon < 9$) to ε of the parallel classes, and adjoin a block of the ε new points. We obtain a GDD containing 7 groups of size 4 and one group of size ε; and blocks of size 4 and 5. Apply Theorem 5.3, to get a KTS$(171+6\varepsilon)$ containing sub-KTS(27). The desired KTS are obtained when $\varepsilon=2$, 3. □

So, at this point, we have

Lemma 5.13. *If $v=1$ or $4 \leqslant v$, and $u \geqslant 4v$, then there is a* KTS$(6u+3)$ *containing a sub*-KTS$(6v+3)$. *If $v=2$ or 3 and $u \geqslant 28+v$, then there is a* KTS$(6u+3)$ *with a sub*-KTS$(6v+3)$.

We now consider the cases $v=2$ and 3 in detail (i.e., the problem of sub-KTS(15) and sub-KTS(21)).

Lemma 5.14. *If there is a* $(v, \{4, 7, 10\})$-PBD *which contains at least one block of size* 7, *then there is a* KTS$(2v+1)$ with a sub-KTS(15).

Proof. Construct a GDD by taking a block of size 7 to be a group, with all other groups of size 1. Give every point weight 2, applying Construction 3.1. Then apply Construction 5.1. □

In a similar fashion, we can prove

Lemma 5.15. *If there exists a* $(v, \{4, 7, 10\})$-PBD, *which contains at least one block of size* 10, *then there is a* KTS$(2v+1)$ *containing a sub*-KTS(21).

Lemma 5.16. *There is a* KTS(105) *which contains sub*-KTS(15) *and sub*-KTS(21).

Proof. Start with a TD(4, 5) and delete three points from some block A. A now contains two points, but all other blocks have size 4 or 5. The groups are of size 3 and 4. Apply Theorem 5.5, obtaining the desired KTS. □

Lemma 5.17. *There exists a* KTS(177) *with a sub*-KTS(15) *and a sub*-KTS(21); *and a* KTS(183) *with a sub*-KTS(21).

Proof. Begin with a resolvable $(28, 4, 1)$-BIBD, and adjoint ε new points, as in Lemma 5.12. Now delete an old point x, and take the blocks through x as groups. We have a GDD, on $27+\varepsilon$ points, with groups of size 3 and 4, and blocks of size 4, 5, and a unique block of size ε. We can apply Theorem 5.5. When $\varepsilon = 2$ we get a KTS(177); when $\varepsilon = 3$, we obtain a KTS(183). These KTSs contain the desired subsystems. □

Lemma 5.18. *For all* $v \equiv 3 \pmod 6$, $v \geqslant 45$, $v \neq 81$, *there is a* KTS(v) *containing a sub*-(KTS(15).

Proof. If $v \geqslant 183$, we are done by Lemma 5.13. If $v \equiv 3 \pmod{12}$, we obtain the desired KTS from a frame of 12^u, applying Construction 5.1 with $\varepsilon = 3$. If $v \equiv 21 \pmod{24}$, then there exists a $(\frac{1}{2}(v-1), \{4, 7\})$-PBD which contains a block of size 7, by Brouwer [4]. Apply Lemma 5.14. We handle $v = 57$ and $v = 129$ also using Lemma 5.14: A TD(4, 7) gives a $(28, \{4, 7\})$-PBD, and adding a point to the groups of a TD(7, 9) gives a $(64, \{7, 10\})$-PBD. Next, $v = 105$ was done in Lemma 5.16 and we handled $v = 177$ in Lemma 5.17. The value $v = 153$ is killed with Corollary 5.7 ($v = 2, n = 5, k = 6$). This leaves $v = 81$ as the only possible exception. □

Lemma 5.19. *There exists a* KTS(99) *containing a sub*-KTS(21).

Proof. Add one point to an affine plane of order 4 and delete another, producing a GDD with groups of size 3 and 4, and blocks of size 4 and 5. Apply Theorem 5.3. □

Lemma 5.20. *There exists a* KTS(123) *which contains a sub*-KTS(21).

Proof. From an affine plane of order 5, delete four points on some line L, and a point x not on L. Taking the blocks through x as groups, we get a GDD with groups of size 3 and 4, and blocks of size 4 and 5. Apply Theorem 5.3. □

Lemma 5.21. *There exists a* KTS(135) *containing a sub*-KTS(21).

Proof. Delete three non-collinear points from an affine plane of order 5. This produces a GDD with groups of size three and four, and blocks of size 4, 5, and one block of size 3. The block of size 3 hits only groups of size 4, so Theorem 5.5 may be applied. □

Table 2
Construction of KTS with sub-KTS(21)

Order	Construction	Remarks
63	Lemma 5.14	A $(31, \{4, 10\})$-PBD is constructed by adding 10 infinite points to a KTS(21)
69	Lemma 5.14	There is a $(34, \{4, 7, 10\})$-PBD which contains a block of size 10, by Wilson [10, Lemma 5.2]
75	Construction 5.1	Frame of type 18^4
81	Lemma 5.14	A TD(4, 10) is a $(40, \{4, 10\})$-PBD
87	??	
93	Construction 5.1	Frame of type 18^5
99	Lemma 5.19	
105	Lemma 5.16	
111	Construction 5.1	Frame of type 18^6
117	Corollary 5.7	$v=3$, $n=4$, $k=5$
123	Lemma 5.20	
129	Construction 5.1	Frame of type 18^7
135	Lemma 5.21	
141–171	Lemma 5.22	
177	Lemma 5.17	
183	Lemma 5.17	

Lemma 5.22. *For* $141 \leqslant u \leqslant 171$, $u \equiv 3 \pmod 6$, *there is a* KTS(u) *which contains a sub*-KTS(21).

Proof. This follows from Corollary 5.7, with $v=3$, $n=5$, and $k=6$. □

We now present constructions for STS(u) containing sub-KTS(21), $u \leqslant 177$, in Table 2. Summarizing, we have

Lemma 5.23. *For* $u \geqslant 63$, $u \neq 87$, $u \equiv 3 \pmod 6$, *there exists a* KTS(u) *containing a sub*-KTS(21).

We gather together all the results of this section to produce

Theorem 5.24. *For* $u, v \equiv 3 \pmod 6$, $v \geqslant 27$, *and* $u \geqslant 4v - 9$, *there is a* KTS(u) *containing a sub*-KTS(v).

For $v = 3$, 9, 15, *or* 21, *and* $u \equiv 3 \pmod 6$, $u \geqslant 3v$, *and* $(v, u) \neq (81, 15)$ *or* $(87, 21)$, *there exists a* KTS(u) *containing a sub*-KTS(v).

Acknowledgments

I would like to thank the referee for providing the short proof of the existence of frames of type 4^u in Lemma 4.1.

References

[1] A. Assaf, E. Mendelsohn and D.R. Stinson, On resolvable coverings of pairs by triples, Utilitas Math., to appear.

[2] R.D. Baker and R.M. Wilson, Nearly Kirkman triple systems, Utilitas Math. 11 (1977) 289–296.

[3] A.E. Brouwer, H. Hanani and A. Schrijver, Group divisible designs with block-size four, Discrete Math. 20 (1977) 1–10.

[4] A.E. Brouwer, Optimal packings of K_4's into a K_n, J. Combin. Theory Ser. A 26 (1979) 278–297.

[5] A.E. Brouwer, The number of mutually orthogonal Latin squares: a table up to order 10000, Research Report ZW 123/79, Math. Centrum, Amsterdam (1979).

[6] J.D. Horton, Sub-Latin squares and incomplete orthogonal arrays. J. Combin. Theory Ser. A 16 (1974) 23–33.

[7] D.K. Ray-Chaudhuri and R.M. Wilson, Solution of Kirkman's school-girl problem, Amer. Math. Soc. Proc. Symp. Pure Math. 19 (1971) 187–204.

[8] R. Rees and D.R. Stinson, On resolvable group-divisible designs with block-size 3, Ars Combin., to appear.

[9] D.R. Stinson, The equivalence of certain incomplete transversal designs and frames, Ars Combin., to appear.

[10] R.M. Wilson, Constructions and uses of pairwise balanced designs, Math. Centre Tracts 55 (1974) 18–41.